$$f(x) = \left[\sqrt{9k+13}\right], \forall x \in \left[\frac{2}{3}, \frac{7}{9}\right]$$

We obtain if we denote

$$S = \left[\sqrt{9k+13}\right] \leq \sqrt{k+x} + \sqrt{k+x+1} + \sqrt{k+x+2} \leq S+1$$

$\forall x \in \left[\frac{2}{3}, \frac{7}{9}\right]$.
After integrate on $\left(\frac{2}{2}, t\right)$, $t \in \left[\frac{2}{3}, \frac{7}{9}\right]$ we obtain:

$$S\left(t - \frac{2}{3}\right) \leq \frac{2}{3}\left\lfloor \sqrt{(x+k)^3} + \sqrt{(x+k+1)^3} + \sqrt{(x+k+2)^3}\right|_{\frac{2}{3}}^{t} <$$

$$< (S+1)\left(t - \frac{2}{3}\right), \ \forall t \in \left[\frac{2}{3}, \frac{7}{9}\right]$$

$$S(3t-2) \leq 2\left(\sqrt{(t+k)^3} + \sqrt{(t+k+1)^3} + \sqrt{t+(k+2)^3} - \right.$$

$$\left. -\sqrt{\left(k+\frac{2}{3}\right)^3} - \sqrt{\left(k+\frac{5}{3}\right)^3} - \sqrt{\left(k+\frac{8}{3}\right)^3}\right) <$$

$$< (S+1)(3t-2), \ \forall t \in \left[\frac{2}{3}, \frac{7}{9}\right]$$

from which it follows the statement.

W44. Solution by the proposer. From

$$x - \frac{x^2}{2} \leq \ln(x+1) \leq \lambda, \ \forall x \geq 0$$

We obtain replacing $x \to \left\{\frac{1}{n}, \frac{1}{n\sqrt{2}}, ..., \frac{1}{n\sqrt{n}}\right\}$. After summing that

$$u_n = \frac{1}{n}\sum_{k-=1}^{n} \frac{1}{\sqrt{k}} - \frac{1}{2}\frac{1}{n^2}\sum_{k-=1}^{n} \frac{1}{k} \leq$$

$$\leq \ln \prod_{k=1}^{n}\left(1 + \frac{1}{n\sqrt{k}}\right) \leq \frac{1}{n}\sum_{k-=1}^{n} \frac{1}{\sqrt{k}} =???$$

or

$$e^{u_n} \leq \prod_{k=1}^{n} \left(1 + \frac{1}{n\sqrt{k}}\right) \leq e^{v_n}$$

where $u_n, v_n \to 0$. So

$$\prod_{k=1}^{n} \left(1 + \frac{1}{n\sqrt{k}}\right) \to 1$$

we have

$$n\left(e^{u_n} - 1\right) - 2\sqrt{n} \leq a_n \leq n\left(e^{v_n} - 1\right) - ??$$

$$a_n = n\left[\left(1 + \frac{1}{u}\right) ... \left(1 + \frac{n}{n\sqrt{n}}\right) - 2\sqrt{???}\right]$$

We compute

$$\lim_{n\to\infty}\left[n\left(e^{u_n} - 1\right) - 2\sqrt{n}\right] = \lim_{n\to\infty}\left[\frac{n\left(e^{u_n} - 1\right)}{u_n} u_n - 2\sqrt{n}\right] =$$

$$= \lim_{n\to\infty}\left\{\frac{e^{u_n - 1}}{u_n}\left[\sum_{k=1}^{n} \frac{1}{\sqrt{k}} - \frac{1}{2}\frac{1}{n}\sum_{k=1}^{n} - ?????\right]\right\} =$$

$$= \lim_{n\to\infty}\left(\frac{e^{u_n} - 1}{u_n}\sum_{k=1}^{n}\frac{1}{\sqrt{k}} - 2\sqrt{n}\right) - \lim_{n\to\infty}\frac{e^{u_n} - 1}{u_n}\frac{1}{2}\frac{1}{n}\sum_{k=1}^{n} =$$

$$= \lim_{n\to\infty}\left[\frac{e^{u_n} - 1}{u_n}\left(b_n + 2\sqrt{n}\right) - 2\sqrt{n}\right] =$$

(where $b_n = \sum\limits_{k=1}^{n} \frac{1}{\sqrt{n}} - 2\sqrt{n}$, represent Ioachimescu sequence with $\lim\limits_{n\to\infty} b_n = \alpha \in (-2, -1)$

$$= \lim_{n\to\infty}\frac{e^{u_n} - 1}{u_n} b_n + \lim_{n\to\infty}\frac{2\sqrt{n}e^{u_n} - 1 - u_n^{\frac{1}{2}}}{u_n^2} u_n =$$

$$= \alpha + \lim_{n\to\infty}\sqrt{n}\frac{1}{n}\left(\sum_{k=1}^{n}\frac{1}{\sqrt{k}} - \frac{1}{2}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k}\right) =$$

$$= \alpha + \lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\frac{1}{\sqrt{k}} - \frac{1}{2n}\lim_{n\to\infty}\frac{1}{n\sqrt{n}}\sum_{k=1}^{n}\frac{1}{k} =$$

$$= c_n = 1 + \frac{1}{n} + ... = \alpha + 2$$

Also

$$\lim_{n\to\infty}\left[n\left(ie^{v_n}-1\right)-2\sqrt{n}\right] = \lim_{n\to\infty}\left[n\frac{e^{v_n}-1}{v_n}v_n - 2\sqrt{n}\right] =$$

$$= \lim_{n\to\infty}\left(\frac{e^{v_n}-1}{v_n}\sum_{k=1}^{n}\frac{1}{\sqrt{k}} - 2\sqrt{n}\right) =$$

$$= \lim_{n\to\infty}\left[\frac{e^{v_n}-1}{v_n}\left(b_n+2\sqrt{n}\right)-2\sqrt{n}\right] =$$

$$= \lim_{n\to\infty}\frac{e^{v_n}-1}{v_n}b_n + \lim_{n\to\infty}2\sqrt{n}\frac{e^{v_n}-1-v_n}{v_n^2}v_n =$$

$$= \alpha + \lim_{n\to\infty}\sqrt{n}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sqrt{k}} = \alpha + 2$$

so

$$\lim_{n\to\infty} a_n = \alpha + 2 \in (0,1)$$

Second solution. Let $P_n := \prod_{k=1}^{n}\left(1+\frac{1}{n\sqrt{k}}\right)$, $S_n := \sum_{k=1}^{n}\frac{1}{\sqrt{k}}$, $H_n := \sum_{k=1}^{n}\frac{1}{k}$ and let

$$\sigma_k = \sigma_k\left(x_1, x_2, ..., x_n\right) := \sum_{1\le i_1<i_2<...<i_k\le n} x_{i_1}x_{i_2}...x_{i_k}$$

for any positive real numbers $x_1, x_2, ..., x_n$, where $k = 1, 2, ..., n$.
Since

$$\frac{1}{\sqrt{k}} < 2\left(\sqrt{k}-\sqrt{k-1}\right)$$

then

$$S_n < \sum_{k=1}^{n} 2\left(\sqrt{k}-\sqrt{k-1}\right) = 2\sqrt{n}$$

Also, noting that $S_n - 2\sqrt{n+1}$ increase and $S_n - 2\sqrt{n}$ decrease by $n \in \mathbb{N}$ we can define some constant

$$c := \lim_{n\to\infty}\left(S_n - 2\sqrt{n}\right) = \lim_{n\to\infty}\left(S_n - 2\sqrt{n+1}\right).$$

Thus, denoting $\alpha_n := S_n - 2\sqrt{n} - c$ we obtain $S_n = 2\sqrt{n} + c + \alpha_n$, where $\lim\limits_{n\to\infty} \alpha_n = 0$.
We have

$$P_n = 1 + \frac{S_n}{n} + \frac{1}{n^2} \sum_{1\le i<j\le n}^{n} \frac{1}{\sqrt{ij}} + R_n,$$

where

$$R_n := \sum_{k=3}^{n} \frac{1}{n^k} \sum_{1\le i_1<i_2<\ldots<i_k j\le n}^{n} \frac{1}{\sqrt{i_1 \cdot i_2 \cdot \ldots \cdot i_k}}$$

Applying Maclaurin's inequality $\sqrt[k]{\frac{\sigma_k}{\binom{n}{k}}} \le \frac{\sigma_1}{n}$ in the form $\sigma_k \le \frac{\binom{n}{k}}{n^k}\sigma_1$ to

$x_n = \frac{1}{\sqrt{n}}, n \in \mathbb{N}$ we obtain

$$\frac{1}{\sqrt{i_1 \cdot i_2 \cdot \ldots \cdot i_k}} \le \frac{\binom{n}{k}}{n^k}\sigma_1^k = \frac{\binom{n}{k}}{n^k} S_n^k$$

and, therefore,

$$R_n \le \sum_{k=3}^{n} \frac{1}{n^k} \cdot \frac{\binom{n}{k}}{n^k} S_n^k.$$

Since $\frac{\binom{n}{k}}{n^k} < \frac{1}{k!}$ and $S_n < 2\sqrt{n}$ then

$$R_n < \sum_{k=3}^{n} \frac{1}{n^k k!} \cdot 2^k n^{k/2} \le \frac{1}{n^{3/2}} \sum_{k=3}^{n} \frac{2^k}{k!} < \frac{e^2}{n^{3/2}}.$$

Thus,

$$0 < P_n - 1 - \frac{S_n}{n} - \frac{1}{n^2} \sum_{1\le i<j\le n}^{n} \frac{1}{\sqrt{ij}} < \frac{e^2}{n^{3/2}} \iff$$

$$\iff 0 < n\,(P_n - 1) - S_n - \frac{1}{n} \sum_{1\le i<j\le n}^{n} \frac{1}{\sqrt{ij}} < \frac{e^2}{n^{1/2}}$$

and, therefore,

$$\lim_{n\to\infty}\left(n\left(P_n-1\right)-S_n-\frac{1}{n}\sum_{1\le i<j\le n}^{n}\frac{1}{\sqrt{ij}}\right)=0.$$

Noting that

$$\sum_{1\le i<j\le n}^{n}\frac{1}{\sqrt{ij}}=\frac{1}{2}\left(\left(\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\right)^2-\sum_{k=1}^{n}\left(\frac{1}{\sqrt{k}}\right)^2\right)=\frac{1}{2}\left(S_n^2-H_n\right)$$

and

$$\lim_{n\to\infty}\frac{H_n}{n}=0,\ \lim_{n\to\infty}\frac{S_n^2}{n}=\lim_{n\to\infty}\frac{\left(2\sqrt{n}+c+\alpha_n\right)^2}{n}=$$

$$=\lim_{n\to\infty}\frac{4n+4\sqrt{n}\left(c+\alpha_n\right)+\left(c+\alpha_n\right)^2}{n}=4$$

we obtain

$$\lim_{n\to\infty}\frac{1}{n}\sum_{1\le i<j\le n}^{n}\frac{1}{\sqrt{ij}}=\frac{1}{2}\lim_{n\to\infty}\frac{1}{n}\left(S_n^2-H_n\right)=2$$

and, therefore,

$$\lim_{n\to\infty}\left(n\left(P_n-1\right)-S_n-\frac{1}{n}\sum_{1\le i<j\le n}^{n}\frac{1}{\sqrt{ij}}\right)=0\iff$$

$$\iff\lim_{n\to\infty}\left(n\left(P_n-1\right)-2\sqrt{n}-c-\alpha_n-\frac{1}{n}\sum_{1\le i<j\le n}^{n}\frac{1}{\sqrt{ij}}\right)=0\iff$$

$$\lim_{n\to\infty}\left(n\left(P_n-1\right)-2\sqrt{n}\right)=c+2$$

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W45. Solution by the proposer. We have:

$$\begin{cases} z_3-z_1=k\left(z_6-z_1\right)i\\ z_4-z_2=k\left(z_1-z_2\right)i\\ z_5-z_3=k\left(z_2-z_3\right)i\\ z_6-z_4=k\left(z_3-z_4\right)i\\ z_1-z_5=k\left(z_4-z_5\right)i \end{cases}$$